

# Super-de Sitter and alternative super-Poincaré symmetries

V.N. Tolstoy

**Abstract** It is well-known that de Sitter Lie algebra  $\mathfrak{o}(1,4)$  contrary to anti-de Sitter one  $\mathfrak{o}(2,3)$  does not have a standard  $\mathbb{Z}_2$ -graded superextension. We show here that the Lie algebra  $\mathfrak{o}(1,4)$  has a superextension based on the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading. Using the standard contraction procedure for this superextension we obtain an *alternative* super-Poincaré algebra with the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading.

## 1 Introduction

In supergravity theory (SUGRA) already for more than 20 years there is the following unsolved (up to now) problem. All physical reasonable solutions of SURGA models with cosmological constants  $\Lambda$  have been constructed for the case  $\Lambda < 0$ , i.e. for the anti-de Sitter metric

$$g_{ab} = \text{diag}(1, -1, -1, -1, 1), \quad (a, b = 0, 1, 2, 3, 4) \quad (1)$$

with the space-time symmetry  $\mathfrak{o}(2,3)$ . In the case  $\Lambda > 0$ , i.e. for the Sitter metric

$$g_{ab} = \text{diag}(1, -1, -1, -1, -1), \quad (a, b = 0, 1, 2, 3, 4) \quad (2)$$

with the space-time symmetry  $\mathfrak{o}(1,4)$  no reasonable solutions have been found. For example, in SUGRA it was obtained the following relation

$$\Lambda = -3m^2, \quad (3)$$

where  $m$  is the massive parameter of gravitinos. Thus if  $\Lambda > 0$ , then  $m$  is imaginary.

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V.N. Tolstoy

Lomonosov Moscow State University, Skobeltsyn Institute of Nuclear Physics (MSU SINP), 1(2) Leninskie Gory, GSP-1, Moscow 119991, Russian Federation, e-mail: [tolstoy@nucl-th.sinp.msu.ru](mailto:tolstoy@nucl-th.sinp.msu.ru)

In my opinion these problems for the case  $\Lambda > 0$  are connected with superextensions of anti-de Sitter  $\mathfrak{o}(2,3)$  and de Sitter  $\mathfrak{o}(1,4)$  symmetries. The  $\mathfrak{o}(2,3)$  symmetry has the superextension - the superalgebra  $\mathfrak{osp}(1|(2,3))$ . This is the usual  $\mathbb{Z}_2$ -graded superalgebra. In the case of  $\mathfrak{o}(1,4)$  such superextension does not exist. However the Lie algebra  $\mathfrak{o}(1,4)$  has an *alternative* superextension that is based on the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading and a preliminary analysis shows that we can construct the reasonable SUGRA models for the case  $\Lambda > 0$ . In this paper we shall consider certain  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded supersymmetries, but we will not discuss supergravity models based on such supersymmetries.

All standard relativistic SUSY (super-anti de Sitter, super-Poincaré, super-conformal, extended  $N$ -supersymmetry, etc) are based on usual ( $\mathbb{Z}_2$ -graded) Lie superalgebras ( $\mathfrak{osp}(1|(2,3))$ ,  $\mathfrak{su}(N|(2,2))$ ,  $\mathfrak{osp}(N|(2,3))$  etc). It turns out that every standard relativistic SUSY has an alternative variant based on an alternative ( $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded) Lie superalgebra:

$$\text{Standard relativistic SUSY} \rightleftharpoons \text{Alternative relativistic SUSY}$$

Distinctive features of the standard and alternative relativistic symmetries (in the example of Poincaré SUSY) are connected with the relations between the four-momenta and the  $Q$ -charges and also between the space-time coordinates and the Grassmann variables. Namely, we have.

(I) *For the standard ( $\mathbb{Z}_2$ -graded) Poincaré SUSY:*

$$[P_\mu, Q_\alpha] = [P_\mu, \bar{Q}_{\dot{\alpha}}] = 0, \quad \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu, \quad (4)$$

$$[x_\mu, \theta_\alpha] = [x_\mu, \bar{\theta}_{\dot{\alpha}}] = \{\theta_\alpha, \bar{\theta}_{\dot{\beta}}\} = 0. \quad (5)$$

(II) *For the alternative ( $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded) Poincaré SUSY:*

$$\{P_\mu, Q_\alpha\} = \{P_\mu, \bar{Q}_{\dot{\alpha}}\} = 0, \quad [Q_\alpha, \bar{Q}_{\dot{\beta}}] = 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu, \quad (6)$$

$$\{x_\mu, \theta_\alpha\} = \{x_\mu, \bar{\theta}_{\dot{\alpha}}\} = [\theta_\alpha, \bar{\theta}_{\dot{\beta}}] = 0. \quad (7)$$

We wrote down only the relations which are changed in the  $\mathbb{Z}_2$ - and  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -cases.

The paper is organized as follows. Section 2 provides definitions and general structure of  $\mathbb{Z}_2$ - and  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superalgebras and also some classification of such simple Lie superalgebras. In Section 3 we describe the orthosymplectic  $\mathbb{Z}_2$ - and  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superalgebras  $\mathfrak{osp}(1|4)$  and  $\mathfrak{osp}(1|2,2)$  and their real forms. We show here that a real form of  $\mathfrak{osp}(1|4)$  contains  $\mathfrak{o}(2,3)$  and a real form of  $\mathfrak{osp}(1|2,2)$  contains  $\mathfrak{o}(1,4)$ . In Section 4 using the standard contraction procedure for the superextension  $\mathfrak{osp}(1|2,2)$  we obtain an *alternative* super-Poincaré algebra with the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading.

## 2 $\mathbb{Z}_2$ - and $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebras

A  $\mathbb{Z}_2$ -graded superalgebra [1]. A  $\mathbb{Z}_2$ -graded Lie superalgebra (LSA)  $\mathfrak{g}$ , as a linear space, is a direct sum of two graded components

$$\mathfrak{g} = \bigoplus_{a=0,1} \mathfrak{g}_a = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad (8)$$

with a bilinear operation (the general Lie bracket),  $[[\cdot, \cdot]]$ , satisfying the identities:

$$\deg([x_a, y_b]) = \deg(x_a) + \deg(y_b) = a + b \pmod{2}, \quad (9)$$

$$[[x_a, y_b]] = -(-1)^{ab} [[y_b, x_a]], \quad (10)$$

$$[[x_a, [y_b, z]]] = [[[x_a, y_b], z]] + (-1)^{ab} [[y_b, [x_a, z]]], \quad (11)$$

where the elements  $x_a$  and  $y_b$  are homogeneous,  $x_a \in \mathfrak{g}_a$ ,  $y_b \in \mathfrak{g}_b$ , and the element  $z \in \mathfrak{g}$  is not necessarily homogeneous. The grading function  $\deg(\cdot)$  is defined for homogeneous elements of the subspaces  $\mathfrak{g}_0$  and  $\mathfrak{g}_1$  modulo 2,  $\deg(\mathfrak{g}_0) = 0$ ,  $\deg(\mathfrak{g}_1) = 1$ . The first identity (9) is called the grading condition, the second identity (10) is called the symmetry property and the condition (11) is the Jacobi identity. It follows from (9) that  $\mathfrak{g}_0$  is a Lie subalgebra in  $\mathfrak{g}$ , and  $\mathfrak{g}_1$  is a  $\mathfrak{g}_0$ -module. It follows from (9) and (10) that the general Lie bracket  $[[\cdot, \cdot]]$  for homogeneous elements posses two values: commutator  $[\cdot, \cdot]$  and anticommutator  $\{\cdot, \cdot\}$ .

A  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superalgebra [4]. A  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded LSA  $\tilde{\mathfrak{g}}$ , as a linear space, is a direct sum of four graded components

$$\tilde{\mathfrak{g}} = \bigoplus_{\mathbf{a}=(a_1, a_2)} \tilde{\mathfrak{g}}_{\mathbf{a}} = \tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)} \oplus \tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)} \quad (12)$$

with a bilinear operation  $[[\cdot, \cdot]]$  satisfying the identities (grading, symmetry, Jacobi):

$$\deg([x_{\mathbf{a}}, y_{\mathbf{b}}]) = \deg(x_{\mathbf{a}}) + \deg(y_{\mathbf{b}}) = \mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2), \quad (13)$$

$$[[x_{\mathbf{a}}, y_{\mathbf{b}}]] = -(-1)^{\mathbf{a}\mathbf{b}} [[y_{\mathbf{b}}, x_{\mathbf{a}}]], \quad (14)$$

$$[[x_{\mathbf{a}}, [y_{\mathbf{b}}, z]]] = [[[x_{\mathbf{a}}, y_{\mathbf{b}}], z]] + (-1)^{\mathbf{a}\mathbf{b}} [[y_{\mathbf{b}}, [x_{\mathbf{a}}, z]]], \quad (15)$$

where the vector  $(a_1 + b_1, a_2 + b_2)$  is defined  $\pmod{2, 2}$  and  $\mathbf{a}\mathbf{b} = a_1 b_1 + a_2 b_2$ . Here in (13)-(15)  $x_{\mathbf{a}} \in \tilde{\mathfrak{g}}_{\mathbf{a}}$ ,  $y_{\mathbf{b}} \in \tilde{\mathfrak{g}}_{\mathbf{b}}$ , and the element  $z \in \tilde{\mathfrak{g}}$  is not necessarily homogeneous. It follows from (13) that  $\tilde{\mathfrak{g}}_{(0,0)}$  is a Lie subalgebra in  $\tilde{\mathfrak{g}}$ , and the subspaces  $\tilde{\mathfrak{g}}_{(1,1)}$ ,  $\tilde{\mathfrak{g}}_{(1,0)}$  and  $\tilde{\mathfrak{g}}_{(0,1)}$  are  $\tilde{\mathfrak{g}}_{(0,0)}$ -modules. It should be noted that  $\tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)}$  is a Lie subalgebra in  $\tilde{\mathfrak{g}}$  and the subspace  $\tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)}$  is a  $\tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)}$ -module, and moreover  $\{\tilde{\mathfrak{g}}_{(1,1)}, \tilde{\mathfrak{g}}_{(1,0)}\} \subset \tilde{\mathfrak{g}}_{(0,1)}$  and vice versa  $\{\tilde{\mathfrak{g}}_{(1,1)}, \tilde{\mathfrak{g}}_{(0,1)}\} \subset \tilde{\mathfrak{g}}_{(1,0)}$ . It follows from (13) and (14) that the general Lie bracket  $[[\cdot, \cdot]]$  for homogeneous elements posses two values: commutator  $[\cdot, \cdot]$  and anticommutator  $\{\cdot, \cdot\}$  as well as in the previous  $\mathbb{Z}_2$ -case.

Let us introduce a useful notation of parity of homogeneous elements: *the parity  $p(x)$  of a homogeneous element  $x$  is a scalar square of its grading  $\deg(x)$  modulo 2*. It is evident that for the  $\mathbb{Z}_2$ -graded superalgebra  $\mathfrak{g}$  the parity coincides with the grading:  $p(\mathfrak{g}_a) = \deg(\mathfrak{g}_a) = \bar{a}$  ( $\bar{a} = \bar{0}, \bar{1}$ )<sup>1</sup>. In the case of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superalgebra  $\tilde{\mathfrak{g}}$  we have

$$p(\tilde{\mathfrak{g}}_a) := \mathbf{a}^2 = a_1^2 + a_2^2 \pmod{2}, \quad (16)$$

that is

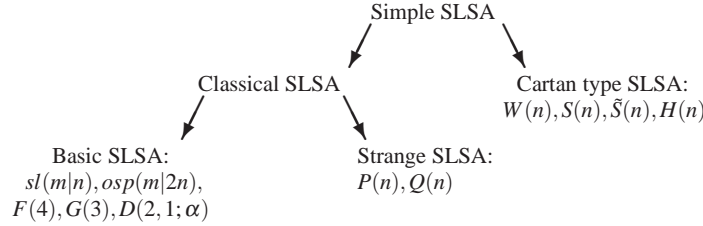
$$p(\tilde{\mathfrak{g}}_{(0,0)}) = p(\tilde{\mathfrak{g}}_{(1,1)}) = \bar{0}, \quad p(\tilde{\mathfrak{g}}_{(1,0)}) = p(\tilde{\mathfrak{g}}_{(0,1)}) = \bar{1}. \quad (17)$$

Homogeneous elements with the parity  $\bar{0}$  are called even and with parity  $\bar{1}$  are odd. Thus,

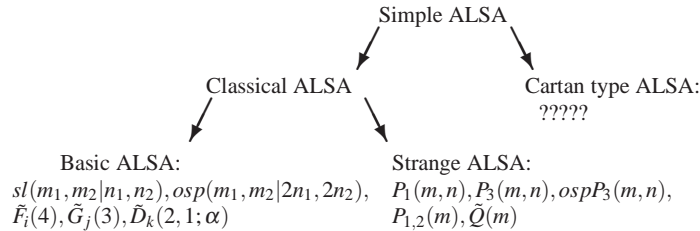
$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{\bar{0}} \oplus \tilde{\mathfrak{g}}_{\bar{1}}, \quad \tilde{\mathfrak{g}}_{\bar{0}} = \tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)}, \quad \tilde{\mathfrak{g}}_{\bar{1}} = \tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)}. \quad (18)$$

The even subspace  $\tilde{\mathfrak{g}}_{\bar{0}}$  is a subalgebra and the odd one  $\tilde{\mathfrak{g}}_{\bar{1}}$  is a  $\tilde{\mathfrak{g}}_{\bar{0}}$ -module. Thus the parity unifies "cousinly" the  $\mathbb{Z}_2$ - and  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superalgebras.

*Classification of the  $\mathbb{Z}_2$ - and  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded simple Lie superalgebras.* A complete list of simple  $\mathbb{Z}_2$ -graded (standard) Lie superalgebras was obtained by Kac [1]. The following scheme resumes the classification [2]:



There is a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -analog (alternative superalgebras) of this scheme:



where  $i = 1, 2, \dots, 6$ ,  $j = 1, 2, 3$ ,  $k = 1, 2, 3$ . It should be noted that the classification of the classical series  $sl(m_1, m_2|n_1, n_2)$ ,  $osp(m_1, m_2|2n_1, 2n_2)$  and all strange series was obtained by Rittenberg and Wyler in [4].

<sup>1</sup> Integer value of the parity will be denoted with the bar.

There are numerous references about the  $\mathbb{Z}_2$ -graded Lie superalgebras and their applications. Unfortunately, in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -case the situation is somewhat poor. There are a few references where some  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebras were studied and applied [3]–[8].

Analysis of matrix realizations of the basic  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebras shows that these superalgebras (as well as the  $\mathbb{Z}_2$ -graded Lie superalgebras) have Cartan-Weyl and Chevalley bases, Weyl groups, Dynkin diagrams, etc. However these structures have a specific characteristics for the  $\mathbb{Z}_2$ - and  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded cases. Let us consider, for example, the Dynkin diagrams. In the case of the  $\mathbb{Z}_2$ -graded superalgebras the nodes of the Dynkin diagram and corresponding simple roots occur at three types:

white  $\bigcirc$ , gray  $\otimes$ , dark  $\bullet$ .

While in the case of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superalgebras we have six types of nodes:

(00)-white  $\bigcirc$ , (11)-white  $\bigcirc$ , (10)-gray  $\otimes$ ,  
(01)-gray  $\otimes$ , (10)-dark  $\bullet$ , (01)-dark  $\bullet$ .

In the next Section we consider in detail two basic superalgebras of rank 2: the orthosymplectic  $\mathbb{Z}_2$ -graded superalgebra  $\mathfrak{osp}(1|4)$  and the orthosymplectic  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superalgebra  $\mathfrak{osp}(1|2,2) := \mathfrak{osp}(1,0|2,2)$ . It will be shown that their real forms, which contain the Lorentz subalgebra  $\mathfrak{o}(1,3)$ , give us the super-anti-de Sitter (in the  $\mathbb{Z}_2$ -graded case) and super-de Sitter (in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded case) Lie superalgebras.

### 3 Anti-de Sitter and de Sitter superalgebras

*The orthosymplectic  $\mathbb{Z}_2$ -graded superalgebra  $\mathfrak{osp}(1|4)$ .* The Dynkin diagram:



The Serre relations:

$$[e_{\pm\alpha}, [e_{\pm\alpha}, e_{\pm\beta}]] = 0, \quad [\{[e_{\pm\alpha}, e_{\pm\beta}], e_{\pm\beta}\}, e_{\pm\beta}] = 0. \quad (19)$$

The root system  $\Delta_+$ :

$$\underbrace{2\beta, 2\alpha+2\beta, \alpha, \alpha+2\beta}_{\deg(\cdot)=0}, \underbrace{\beta, \alpha+\beta}_{\deg(\cdot)=1}. \quad (20)$$

*The orthosymplectic  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superalgebra  $\mathfrak{osp}(1|2,2)$ .* The Dynkin diagram:



The Serre relations:

$$\{e_{\pm\alpha}, \{e_{\pm\alpha}, e_{\pm\beta}\}\} = 0, \quad \{[\{e_{\pm\alpha}, e_{\pm\beta}\}, e_{\pm\beta}], e_{\pm\beta}\} = 0. \quad (21)$$

The root system  $\Delta_+$ :

$$\underbrace{2\beta, 2\alpha+2\beta}_{\deg(\cdot)=(00)}, \underbrace{\alpha, \alpha+2\beta}_{\deg(\cdot)=(11)}, \underbrace{\beta}_{\deg(\cdot)=(10)}, \underbrace{\alpha+\beta}_{\deg(\cdot)=(01)}. \quad (22)$$

Commutation relations, which contain Cartan elements, are the same for the  $\mathfrak{osp}(1|4)$  and  $\mathfrak{osp}(1|2, 2)$  superalgebras and they are:

$$\begin{aligned} [[e_\gamma, e_{-\gamma'}]] &= \delta_{\gamma, \gamma'} h_\gamma, \\ [h_\gamma, e_{\gamma'}] &= (\gamma, \gamma') e_{\gamma'} \end{aligned} \quad (23)$$

for  $\gamma, \gamma' \in \{\alpha, \beta\}$ . These relations together with the Serre relations (19) and (21) correspondingly are called the defining relations of the superalgebras  $\mathfrak{osp}(1|4)$  and  $\mathfrak{osp}(1|2, 2)$  correspondingly. It is easy to see that these defining relations are invariant with respect to the non-graded Cartan involution  $(^\dagger)$  ( $(x^\dagger)^\dagger = x$ ,  $[[x, y]]^\dagger = [[y^\dagger, x^\dagger]]$  for any homogenous elements  $x$  and  $y$ ):

$$e_{\pm\gamma}^\dagger = e_{\mp\gamma}, \quad h_\gamma^\dagger = h_\gamma. \quad (24)$$

The composite root vectors  $e_{\pm\gamma}$  ( $\gamma \in \Delta_+$ ) for  $\mathfrak{osp}(1|4)$  and  $\mathfrak{osp}(1|2, 2)$  are defined as follows

$$\begin{aligned} e_{\alpha+\beta} &:= [[e_\alpha, e_\beta]], & e_{\alpha+2\beta} &:= [[e_{\alpha+\beta}, e_\beta]], \\ e_{2\alpha+2\beta} &:= \frac{1}{\sqrt{2}} \{e_{\alpha+\beta}, e_{\alpha+\beta}\}, & e_{2\beta} &:= \frac{1}{\sqrt{2}} \{e_\beta, e_\beta\}, \\ e_{-\gamma} &:= e_\gamma^\dagger. \end{aligned} \quad (25)$$

These root vectors satisfy the non-vanishing relations:

$$\begin{aligned}
[e_\alpha, e_{\alpha+2\beta}] &= (-1)^{\deg \alpha \cdot \deg \beta} \sqrt{2} e_{2\alpha+2\beta}, & [e_\alpha, e_{2\beta}] &= \sqrt{2} e_{\alpha+2\beta}, \\
[[e_{\alpha+\beta}, e_{-\alpha}]] &= -(-1)^{\deg \alpha \cdot \deg \beta} e_\beta, & [e_{\alpha+2\beta}, e_{-\alpha}] &= -\sqrt{2} e_{2\beta}, \\
[e_{2\alpha+2\beta}, e_{-\alpha}] &= -(-1)^{\deg \alpha \cdot \deg \beta} \sqrt{2} e_{\alpha+2\beta}, & [e_{2\beta}, e_{-\beta}] &= -\sqrt{2} e_\beta, \\
[[e_{\alpha+2\beta}, e_{-\alpha-\beta}]] &= -(-1)^{\deg \alpha \cdot \deg \beta} e_\beta, & [[e_\beta, e_{-\alpha-\beta}]] &= e_{-\alpha}, \\
[[e_\beta, e_{-\alpha-2\beta}]] &= -e_{-\alpha-\beta}, & [e_{2\alpha+2\beta}, e_{-\alpha-\beta}] &= -\sqrt{2} e_{\alpha+\beta}, \\
[e_{\alpha+2\beta}, e_{-2\alpha-2\beta}] &= -(-1)^{\deg \alpha \cdot \deg \beta} \sqrt{2} e_{-\alpha}, & [e_{2\beta}, e_{-\alpha-2\beta}] &= -\sqrt{2} e_{-\alpha}, \\
\{e_{\alpha+\beta}, e_{-\alpha-\beta}\} &= h_\alpha + h_\beta, & [e_{\alpha+2\beta}, e_{-\alpha-2\beta}] &= -h_\alpha - 2h_\beta, \\
[e_{2\beta}, e_{-2\beta}] &= -2h_\beta, & [e_{2\alpha+2\beta}, e_{-2\alpha-2\beta}] &= -2h_\alpha - 2h_\beta.
\end{aligned} \tag{26}$$

The rest of non-zero relations is obtained by applying the operation  $(^\dagger)$  to these relations.

Now we find real forms of  $\mathfrak{osp}(1|4)$  and  $\mathfrak{osp}(1|2,2)$ , which contain the real Lorentz subalgebra  $\mathfrak{so}(1,3)$ . It is not difficult to check that the antilinear mapping  $(^*) ((x^*)^* = x, [[x, y]]^* = [[y^*, x^*]])$  for any homogenous elements  $x$  and  $y$ ) given by

$$\begin{aligned}
e_{\pm\alpha}^* &= -(-1)^{\deg \alpha \cdot \deg \beta} e_{\mp\alpha}, & e_{\pm\beta}^* &= -ie_{\pm(\alpha+\beta)}, \\
e_{\pm 2\beta}^* &= -e_{\pm(2\alpha+2\beta)}, & e_{\pm(\alpha+2\beta)}^* &= -e_{\pm(\alpha+2\beta)}, \\
h_\alpha^* &= h_\alpha, & h_\beta^* &= -h_\alpha - h_\beta.
\end{aligned} \tag{27}$$

is an antiinvolution and the desired real form with respect to the antiinvolution is presented as follows.

*The Lorentz algebra  $\mathfrak{o}(1,3)$ :*

$$\begin{aligned}
L_{12} &= -\frac{1}{2} h_\alpha, \\
L_{13} &= -\frac{i}{2\sqrt{2}} (e_{2\beta} + e_{2\alpha+2\beta} + e_{-2\beta} + e_{-2\alpha-2\beta}), \\
L_{23} &= -\frac{1}{2\sqrt{2}} (e_{2\beta} - e_{2\alpha+2\beta} - e_{-2\beta} + e_{-2\alpha-2\beta}), \\
L_{01} &= \frac{i}{2\sqrt{2}} (e_{2\beta} + e_{2\alpha+2\beta} - e_{-2\beta} - e_{-2\alpha-2\beta}), \\
L_{02} &= \frac{1}{2\sqrt{2}} (e_{2\beta} - e_{2\alpha+2\beta} + e_{-2\beta} - e_{-2\alpha-2\beta}), \\
L_{03} &= -\frac{i}{2} (h_\alpha + 2h_\beta).
\end{aligned} \tag{28}$$

The generators  $L_{\mu 4}$ :

$$\begin{aligned}
L_{04} &= -\frac{i}{2} \left( e_{\alpha+2\beta} + (-1)^{\deg \alpha \cdot \deg \beta} e_{-\alpha-2\beta} \right), \\
L_{14} &= -\frac{i}{2} \left( e_{\alpha} + (-1)^{\deg \alpha \cdot \deg \beta} e_{-\alpha} \right), \\
L_{24} &= \frac{1}{2} \left( e_{\alpha} - (-1)^{\deg \alpha \cdot \deg \beta} e_{-\alpha} \right), \\
L_{34} &= -\frac{i}{2} \left( e_{\alpha+2\beta} - (-1)^{\deg \alpha \cdot \deg \beta} e_{-\alpha-2\beta} \right).
\end{aligned} \tag{29}$$

Here are:  $\deg \alpha = 0, \deg \beta = 1$ , i.e.  $(-1)^{\deg \alpha \cdot \deg \beta} = 1$ , for the case of the  $\mathbb{Z}_2$ -grading;  $\deg \alpha = (1, 1), \deg \beta = (1, 0)$ , i.e.  $(-1)^{\deg \alpha \cdot \deg \beta} = -1$ , for the case of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading.

The all elements  $L_{ab}$  ( $a, b = 0, 1, 2, 3, 4$ ) satisfy the relations

$$\begin{aligned}
[L_{ab}, L_{cd}] &= i(g_{bc}L_{ad} - g_{bd}L_{ac} + g_{ad}L_{bc} - g_{ac}L_{bd}), \\
L_{ab} &= -L_{ba}, \quad L_{ab}^* = L_{ab},
\end{aligned} \tag{30}$$

where the metric tensor  $g_{ab}$  is given by

$$\begin{aligned}
g_{ab} &= \text{diag}(1, -1, -1, -1, g_{44}^{(\alpha)}), \\
g_{44}^{(\alpha)} &= (-1)^{\deg \alpha \cdot \deg \beta}.
\end{aligned} \tag{31}$$

Thus we see that in the case of the  $\mathbb{Z}_2$ -grading,  $(-1)^{\deg \alpha \cdot \deg \beta} = 1$ , the generators (28) and (29) generate the anti-de-Sitter algebra  $\mathfrak{o}(2, 3)$ , and in the case of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading,  $(-1)^{\deg \alpha \cdot \deg \beta} = -1$ , the generators (28) and (29) generate the de-Sitter algebra  $\mathfrak{o}(1, 4)$ .

Finally we introduce the "supercharges":

$$\begin{aligned}
Q_1 &:= \sqrt{2} \exp\left(-\frac{i\pi}{4}\right) e_{\alpha+\beta}, & Q_2 &:= \sqrt{2} \exp\left(-\frac{i\pi}{4}\right) e_{-\alpha-\beta}, \\
\bar{Q}_1 &:= \sqrt{2} \exp\left(-\frac{i\pi}{4}\right) e_{\beta}, & \bar{Q}_2 &:= \sqrt{2} \exp\left(-\frac{i\pi}{4}\right) e_{-\beta}.
\end{aligned} \tag{32}$$

They have the following commutation relations between themselves:



$$\begin{aligned}
\{Q_1, Q_1\} &= -i2\sqrt{2}e_{2\alpha+2\beta} = 2(L_{13} - iL_{23} - L_{01} + iL_{02}), \\
\{Q_2, Q_2\} &= -i2\sqrt{2}e_{-2\alpha-2\beta} = 2(L_{13} + iL_{23} - L_{01} - iL_{02}), \\
\{Q_1, Q_2\} &= -i2(h_\alpha + h_\beta) = 2(L_{03} + iL_{12}),
\end{aligned} \tag{33}$$

$$\{\bar{Q}_{\dot{\eta}}, \bar{Q}_{\dot{\zeta}}\} = \{Q_{\dot{\zeta}}, Q_{\dot{\eta}}\}^* \quad (\bar{Q}_{\dot{\eta}} = Q_{\dot{\eta}}^* \text{ for } \eta = 1, 2; \dot{\eta} = \dot{1}, \dot{2}),$$

$$\begin{aligned}
[[Q_1, \bar{Q}_{\dot{1}}]] &= -i2e_{\alpha+2\beta} = 2(L_{04} + L_{34}), \\
[[Q_1, \bar{Q}_{\dot{2}}]] &= -i2e_{\alpha} = 2(L_{14} - iL_{24}), \\
[[Q_2, \bar{Q}_{\dot{1}}]] &= -i2(-1)^{\deg \alpha \deg \beta} e_{-\alpha} = 2(L_{14} + iL_{24}), \\
[[Q_2, \bar{Q}_{\dot{2}}]] &= -i2(-1)^{\deg \alpha \deg \beta} e_{-\alpha-2\beta} = 2(L_{04} - L_{34}).
\end{aligned} \tag{34}$$

Here  $[[\cdot, \cdot]] \equiv \{\cdot, \cdot\}$  for the  $\mathbb{Z}_2$ -case and  $[[\cdot, \cdot]] \equiv [\cdot, \cdot]$  for the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -case. Using the explicit formulas (28), (29), (32) and the commutation relations (26) we can also calculate commutation relations between the operators  $L_{ab}$  and the supercharges  $Q$ 's and  $\bar{Q}$ 's.

#### 4 $\mathbb{Z}_2$ - and $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Poincaré superalgebras

Using the standard contraction procedure:  $L_{\mu 4} = RP_{\mu}$  ( $\mu = 0, 1, 2, 3$ ),  $Q_{\alpha} \rightarrow \sqrt{R} Q_{\alpha}$  and  $\bar{Q}_{\dot{\alpha}} \rightarrow \sqrt{R} \bar{Q}_{\dot{\alpha}}$  ( $\alpha = 1, 2$ ;  $\dot{\alpha} = \dot{1}, \dot{2}$ ) for  $R \rightarrow \infty$  we obtain the super-Poincaré algebra (standard and alternative) which is generated by  $L_{\mu\nu}$ ,  $P_{\mu}$ ,  $Q_{\alpha}$ ,  $\bar{Q}_{\dot{\alpha}}$  where  $\mu, \nu = 0, 1, 2, 3$ ;  $\alpha = 1, 2$ ;  $\dot{\alpha} = \dot{1}, \dot{2}$ , with the relations (we write down only those which are distinguished in the  $\mathbb{Z}_2$ - and  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -cases).

(I) *For the  $\mathbb{Z}_2$ -graded Poincaré SUSY:*

$$[P_{\mu}, Q_{\alpha}] = [P_{\mu}, \bar{Q}_{\dot{\alpha}}] = 0, \quad \{Q_{\alpha}, \bar{Q}_{\dot{\beta}}\} = 2\sigma_{\alpha\dot{\beta}}^{\mu} P_{\mu}. \tag{35}$$

(II) *For the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Poincaré SUSY:*

$$\{P_{\mu}, Q_{\alpha}\} = \{P_{\mu}, \bar{Q}_{\dot{\alpha}}\} = 0, \quad [Q_{\alpha}, \bar{Q}_{\dot{\beta}}] = 2\sigma_{\alpha\dot{\beta}}^{\mu} P_{\mu}, \tag{36}$$

Let us consider the supergroups associated to the  $\mathbb{Z}_2$ - and  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Poincaré superalgebras. A group element  $g$  is given by the exponential of the super-Poincaré generators, namely

$$g(x^{\mu}, \omega^{\mu\nu}, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}}) = \exp(x^{\mu} P_{\mu} + \omega^{\mu\nu} M_{\mu\nu} + \theta^{\alpha} Q_{\alpha} + \bar{Q}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}). \tag{37}$$

Because the grading of the exponent is zero ((0) or (00)) and the result is as follows.

1).  $\mathbb{Z}_2$ -case:  $\deg P = \deg x = 0$ ,  $\deg Q = \deg \bar{Q} = \deg \theta = \deg \bar{\theta} = 1$ . This means that

$$[x_\mu, \theta_\alpha] = [x_\mu, \bar{\theta}_{\dot{\alpha}}] = \{\theta_\alpha, \bar{\theta}_{\dot{\beta}}\} = \{\theta_\alpha, \theta_\beta\} = \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = 0. \quad (38)$$

2).  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -case:  $\deg P = \deg x = (11)$ ,  $\deg Q = \deg \theta = (10)$ ,  $\deg \bar{Q} = \deg \bar{\theta} = (01)$ . This means that

$$\{x_\mu, \theta_\alpha\} = \{x_\mu, \bar{\theta}_{\dot{\alpha}}\} = [\theta_\alpha, \bar{\theta}_{\dot{\beta}}] = \{\theta_\alpha, \theta_\beta\} = \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = 0. \quad (39)$$

One defines the superspaces as the coset spaces of the standard and alternative super-Poincaré groups by the Lorentz subgroup, parameterized the coordinates  $x^\mu$ ,  $\theta^\alpha$ ,  $\bar{\theta}^{\dot{\alpha}}$ , subject to the condition  $\bar{\theta}^{\dot{\alpha}} = (\theta^\alpha)^*$ . We can define a superfield  $\mathcal{F}$  as a function of superspace.

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